# Interactions of Nonlinear Waves in Fluid-Filled Elastic Tubes

Hilmi Demiray

Department of Mathematics, Faculty of Arts and Science, Isik University, Büyükdere Caddesi, 34398 Maslak, Istanbul, Turkey

Reprint requests to H. D.; E-mail: demiray@isikun.edu.tr

Z. Naturforsch. **62a,** 21 – 28 (2007); received November 11, 2006

In this work, treating an artery as a prestressed thin-walled elastic tube and the blood as an inviscid fluid, the interactions of two nonlinear waves propagating in opposite directions are studied in the longwave approximation by use of the extended PLK (Poincaré-Lighthill-Kuo) perturbation method. The results show that up to $O(k^3)$, where $k$ is the wave number, the head-on collision of two solitary waves is elastic and the solitary waves preserve their original properties after the interaction. The leading-order analytical phase shifts and the trajectories of two solitons after the collision are derived explicitly.

*Key words:* Elastic Tubes; Solitary Waves; Wave Interaction.

## 1. Introduction

A remarkable feature of arterial blood flow is its pulsatile character. The intermittent ejection of blood from the left ventricle produces pressure and flow pulses in the arterial tree. Experimental studies reveal that the flow velocity in blood vessels largely depends on the elastic properties of the vessel wall, and that they propagate towards the periphery with a characteristic pattern [1]. Theoretical investigations for the blood waves have been developed by several researchers [2 – 5] through the use of weakly nonlinear theories. It is shown that the dynamics of the blood waves are governed by perturbed Korteweg-de Vries (KdV) equations and their modified forms. The effects of higher-order approximations in a fluid-filled elastic tube with stenosis was studied by us [6] and the KdV equation with variable coefficients was obtained for the first-order term in the perturbation expansion, and the linearized KdV equation with a non-homogeneous term was obtained for the second-order term in the expansion. The solitary wave model gives a plausible explanation for the peaking and steepening of the pulsatile waves in arteries.

In the process of solitary wave propagation in arteries, the wave-wave interactions is another fascinating feature of solitary wave phenomena because the collision of solitary waves exhibits many particle-like features. The unique effect due to the collision is their phase shifts. There are two distinct solitary wave interactions; one is the overtaking collision and the other is the head-on collision [7]. The overtaking collision of solitary waves can be investigated within the context of multisoliton solutions of the KdV equation. The head-on collision between two solitary waves must be studied by a suitable asymptotic expansion to solve the original fluid and tube equations. For such solitary waves it is convenient to use the extended Poincaré-Lighthill-Kuo (PLK) method [7, 8].

In the present work, treating the arteries as prestressed thin-walled elastic tubes and the blood as an inviscid fluid, and utilizing the extended PLK method, we studied the interaction of two weakly nonlinear waves, propagating in opposite directions, in the longwave approximation. The results show that, up to $O(k^3)$, the head-on collision of two solitary waves is elastic and the solitary waves preserve their original properties after the interaction. The leading-order analytical phase shifts and the trajectories of two solitons after the collision are derived explicitly.

## 2. Basic Equations and Theoretical Preliminaries

### 2.1. Equation of the Tube

In this section we shall derive the basic equations governing the motion of a prestressed thin elastic tube filled with an inviscid fluid. For that purpose, we consider a cylindrical long thin tube of radius $R_0$, which is subjected to a uniform inner pressure $P_0$ and an axial stretch ratio $\lambda_z$. Let the radius of the cylindrical tube after such an axially symmetric finite deformation be denoted by $r_0$. Upon this static deformation we superimpose a finite, time-dependent radial displacement $u^*(z^*,t^*)$. The effect of axial displacement will be neglected. Thus, the position vector of a generic point on the tube wall may be represented by

$$\boldsymbol{r} = (r_0 + u^*)\boldsymbol{e_r} + z^*\boldsymbol{e_z}, \quad z^* = \lambda_z Z^*. \tag{1}$$

$\boldsymbol{e_r}$, $\boldsymbol{e_\theta}$ and $\boldsymbol{e_z}$ are the unit base vectors in cylindrical polar coordinates, $Z^*$ is the axial coordinate of a material point in the undeformed configuration and $z^*$ is the spatial coordinate after finite deformation.

The unit tangent vector $\boldsymbol{t}$ to the meridional curve and the unit exterior normal $\boldsymbol{n}$ to the deformed membrane are given by

$$\boldsymbol{t} = \frac{1}{\Lambda}\left(\frac{\partial u^*}{\partial z^*}\boldsymbol{e_r} + \boldsymbol{e_z}\right), \quad \boldsymbol{n} = \frac{1}{\Lambda}\left(\boldsymbol{e_r} - \frac{\partial u^*}{\partial z^*}\boldsymbol{e_z}\right), \tag{2}$$

where $\Lambda$ is defined by

$$\Lambda = [1 + (\frac{\partial u^*}{\partial z^*})^2]^{1/2}. \tag{3}$$

The principal stretch ratios in the meridional and circumferential directions may be given by

$$\lambda_1 = \Lambda\lambda_z, \quad \lambda_2 = \lambda_\theta + u^*/R_0, \tag{4}$$

where $\lambda_\theta = r_0/R_0$ is the initial stretch ratio in the circumferential direction.

Let $T_1$ and $T_2$ be the membrane forces acting along the meridional and circumferential directions, respectively. Then, the equation of radial motion of a small tube element placed between the planes $z^* = \text{const.}$, $z^* + \mathrm{d}z^* = \text{const.}$, $\theta = \text{const.}$ and $\theta + \mathrm{d}\theta = \text{const.}$ is given by

$$-T_2\Lambda + \frac{\partial}{\partial z^*}\left[\frac{(r_0+u^*)}{\Lambda}\frac{\partial u^*}{\partial z^*}T_1\right] + P^*(r_0+u^*) = \rho_0\frac{HR_0}{\lambda_z}\frac{\partial^2 u^*}{\partial t^{*2}}, \tag{5}$$

where $\rho_0$ is the mass density of the tube, $P^*$ is the total fluid pressure and $H$ is the initial thickness of the tube. Here, to be consistent with soft biological tissues, we assume that the tube material is incompressible.

Let $\mu\Sigma(\lambda_1,\lambda_2)$ be the strain energy density function of the tube material, where $\mu$ is the shear modulus of the tube material. Then the membrane forces are expressed as

$$T_1 = \mu\frac{H}{\lambda_2}\frac{\partial\Sigma}{\partial\lambda_1}, \quad T_2 = \mu\frac{H}{\lambda_1}\frac{\partial\Sigma}{\partial\lambda_2}. \tag{6}$$

Introducing (6) into (5) the equation of motion in the radial direction reads

$$\begin{aligned} P^* = {} & \frac{\mu H}{\lambda_z(r_0+u^*)}\frac{\partial\Sigma}{\partial\lambda_2} \\ & - \frac{\mu R_0 H}{(r_0+u^*)}\frac{\partial}{\partial z^*}\left\{\frac{\partial u^*/\partial z^*}{[1+(\partial u^*/\partial z^*)^2]^{1/2}}\frac{\partial\Sigma}{\partial\lambda_1}\right\} \\ & + \frac{\rho_0 H R_0}{\lambda_z(r_0+u^*)}\frac{\partial^2 u^*}{\partial t^{*2}}. \end{aligned} \tag{7}$$

This gives the relation between the radial displacement and the inner pressure applied by the fluid.

### 2.2. Equations of the Fluid

Although blood is known to be an incompressible viscous fluid, as pointed out by Rudinger [9] in a number of applications, e. g. flow in large blood vessels, as a first approximation, the effect of viscosity may be neglected. As a result of this simplifying assumption, the variation of the field quantities in the radial direction may be disregarded. However, the radial changes are included by taking the variation of the cross-sectional area into account. The equation of mass conservation of the fluid may be given by

$$\frac{\partial A^*}{\partial t^*} + \frac{\partial}{\partial z^*}(A^* w^*) = 0, \tag{8}$$

where $A^*$ stands for the cross-sectional area of the tube and $w^*$ is the axial velocity of the fluid. Recalling the definition of $A^*$ in terms of the inner radius of the tube, i. e., $A^* = \pi(r_0+u^*)^2$, (8) may be written as

$$2\frac{\partial u^*}{\partial t^*} + (r_0+u^*)\frac{\partial w^*}{\partial z^*} + 2w^*\frac{\partial u^*}{\partial z^*} = 0. \tag{9}$$

The equation of balance of the linear momentum in the axial direction is given by

$$\frac{\partial w^*}{\partial t^*} + w^*\frac{\partial w^*}{\partial z^*} + \frac{1}{\rho_\mathrm{f}}\frac{\partial P^*}{\partial z^*} = 0, \tag{10}$$

where $\rho_\mathrm{f}$ is the mass density of the fluid body.

At this stage, it is convenient to introduce the following non-dimensionalized quantities:

$$t^* = \frac{R_0}{v_0}t, \quad z^* = R_0 z, \quad u^* = R_0 u, \quad w^* = v_0 w,$$

$$m = \frac{H\rho_0}{R_0\rho_f}, \quad v_0^2 = \frac{\mu H}{\rho_f R_0}, \quad P^* = \rho_f v_0^2(p_0 + p), \tag{11}$$

where $v_0$ is the Moens-Korteweg wave speed. Introducing (11) into (7), (9) and (10), we obtain

$$\begin{aligned} p_0 + p = {} & \frac{1}{\lambda_z(\lambda_\theta + u)} \frac{\partial \Sigma}{\partial \lambda_2} \\ & - \frac{1}{(\lambda_\theta + u)} \frac{\partial}{\partial z} \left\{ \frac{\partial u/\partial z}{[1 + (\partial u/\partial z)^2]^{1/2}} \frac{\partial \Sigma}{\partial \lambda_1} \right\} \\ & + \frac{m}{\lambda_z(\lambda_\theta + u)} \frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{12}$$

$$2\frac{\partial u}{\partial t} + (\lambda_\theta + u)\frac{\partial w}{\partial z} + 2w\frac{\partial u}{\partial z} = 0, \tag{13}$$

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = 0. \tag{14}$$

For our future purposes we need the quadratic series expansion of the pressure in terms of the radial displacement $u$. If this is done, from (12) we have

$$\begin{aligned} p = {} & \beta_1 u + \beta_2 u^2 - \alpha_0 \frac{\partial^2 u}{\partial z^2} - \alpha_1 \left(\frac{\partial u}{\partial z}\right)^2 \\ & + \left(\frac{\alpha_0}{\lambda_\theta} - 2\alpha_1\right) u \frac{\partial^2 u}{\partial z^2} + \frac{m}{\lambda_\theta \lambda_z} \frac{\partial^2 u}{\partial t^2} \\ & - \frac{m}{\lambda_\theta^2 \lambda_z} u \frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{15}$$

where the coefficients $\alpha_0$, $\alpha_1$, $\beta_1$ and $\beta_2$ are defined by

$$\begin{aligned} & \alpha_0 = \frac{1}{\lambda_\theta} \frac{\partial \Sigma}{\partial \lambda_1}|_{u=0}, \quad \alpha_1 = \frac{1}{2\lambda_\theta} \frac{\partial^2 \Sigma}{\partial \lambda_1 \partial \lambda_2}|_{u=0}, \\ & \beta_1 = \left( \frac{1}{\lambda_\theta \lambda_z} \frac{\partial^2 \Sigma}{\partial \lambda_1^2} - \frac{1}{\lambda_\theta^2 \lambda_z} \frac{\partial \Sigma}{\partial \lambda_1} \right)|_{u=0}, \\ & \beta_2 = \frac{1}{2\lambda_\theta \lambda_z} \frac{\partial^3 \Sigma}{\partial \lambda_1^3}|_{u=0} - \frac{\beta_1}{\lambda_\theta}. \end{aligned} \tag{16}$$

Equations (13), (14) and (15) give sufficient relations to determine the unknowns $u$, $w$ and $p$.

## 3. Interaction of Solitary Waves in Blood Vessels

We shall assume that two solitons A and B, which are asymptotically apart from each other in the initial state, travel toward each other. After some time they collide and then depart. In order to analyze the effect of the collision, we shall employ the extended PLK perturbation method with the technique of strained coordinates. According to this method we introduce the following stretched coordinates:

$$\begin{aligned} & \varepsilon^{1/2}(z - c_A t) = \xi - \varepsilon\theta(\xi, \eta), \\ & \varepsilon^{1/2}(z + c_B t) = \eta - \varepsilon\phi(\xi, \eta), \\ & \varepsilon = k^2, \end{aligned} \tag{17}$$

where $k$ is the wave number, $c_A$ and $c_B$ are two constants to be determined from the solution, and $\theta(\xi, \eta)$ and $\phi(\xi, \eta)$ are two unknown phase functions to be determined from the solution.

We shall assume that the constants $c_A$, $c_B$ and the functions $\theta$ and $\phi$ can be expanded into asymptotic series in $k$ as

$$\begin{aligned} & c_A = c_0 + \varepsilon c_{A1} + ..., \quad c_B = c_0 + \varepsilon c_{B1} + ..., \\ & \theta = \theta_0(\eta) + \varepsilon\theta_1(\xi, \eta) + \dots, \\ & \phi = \phi_0(\xi) + \varepsilon\phi_1(\xi, \eta) + \dots. \end{aligned} \tag{18}$$

Then, the partial derivatives $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial t}$ can be expressed as follows:

$$\begin{aligned} \frac{\partial}{\partial z} = {} & \varepsilon^{1/2}\left(1 + \varepsilon\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}\right)\frac{\partial}{\partial \xi} + \varepsilon^{1/2}\left(1 + \varepsilon\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}\right)\frac{\partial}{\partial \eta}, \\ \frac{\partial}{\partial t} = {} & \varepsilon^{1/2}\left[-c_0 + \varepsilon\left(-c_{A1} + c_0\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}\right)\right]\frac{\partial}{\partial \xi} \\ & + \varepsilon^{1/2}\left[c_0 + \varepsilon\left(c_{B1} - c_0\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}\right)\right]\frac{\partial}{\partial \eta}. \end{aligned} \tag{19}$$

We shall further assume that the field variables $u$, $w$ and $p$ may be expressed as asymptotic series in $\varepsilon$ as follows:

$$\begin{aligned} & u = \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \quad w = \varepsilon w_1 + \varepsilon^2 w_2 + \dots, \\ & p = \varepsilon p_1 + \varepsilon^2 p_2 + \dots. \end{aligned} \tag{20}$$

Introducing the expansions (19) and (20) into (13)–(15) and setting the coefficients of like powers of $k$ equal to zero, the following sets of differential equations are obtained:

*$O(\varepsilon)$ equations:*

$$\begin{aligned} & -2c_0\frac{\partial u_1}{\partial \xi} + 2c_0\frac{\partial u_1}{\partial \eta} + \lambda_\theta\left(\frac{\partial w_1}{\partial \xi} + \frac{\partial w_1}{\partial \eta}\right) = 0, \\ & -c_0\frac{\partial w_1}{\partial \xi} + c_0\frac{\partial w_1}{\partial \eta} + \frac{\partial p_1}{\partial \xi} + \frac{\partial p_1}{\partial \eta} = 0, \\ & p_1 = \beta_1 u_1. \end{aligned} \tag{21}$$

*$O(\varepsilon^2)$ equations:*

$$\begin{aligned}
&-2c_0\frac{\partial u_2}{\partial\xi}+2c_0\frac{\partial u_2}{\partial\eta}+\lambda_\theta\left(\frac{\partial w_2}{\partial\xi}+\frac{\partial w_2}{\partial\eta}\right)+2\left(-c_{A1}+c_0\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}\right)\frac{\partial u_1}{\partial\xi}+2\left(c_{B1}-c_0\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}\right)\frac{\partial u_1}{\partial\eta}\\
&\quad+\lambda_\theta\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}\frac{\partial w_1}{\partial\xi}+\lambda_\theta\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}\frac{\partial w_1}{\partial\eta}+u_1\left(\frac{\partial w_1}{\partial\xi}+\frac{\partial w_1}{\partial\eta}\right)+2w_1\left(\frac{\partial u_1}{\partial\xi}+\frac{\partial u_1}{\partial\eta}\right)=0,\\
&-c_0\frac{\partial w_2}{\partial\xi}+c_0\frac{\partial w_2}{\partial\eta}+\frac{\partial p_2}{\partial\xi}+\frac{\partial p_2}{\partial\eta}+\left(-c_{A1}+c_0\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}\right)\frac{\partial w_1}{\partial\xi}+\left(c_{B1}-c_0\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}\right)\frac{\partial w_1}{\partial\eta}+\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}\frac{\partial p_1}{\partial\xi}+\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}\frac{\partial p_1}{\partial\eta}\\
&\quad+w_1\left(\frac{\partial w_1}{\partial\xi}+\frac{\partial w_1}{\partial\eta}\right)=0,\\
&p_2=\beta_1u_2+\beta_2u_1^2-\alpha_0\left(\frac{\partial^2u_1}{\partial\xi^2}+2\frac{\partial^2u_1}{\partial\xi\partial\eta}+\frac{\partial^2u_1}{\partial\eta^2}\right)+\frac{mc_0^2}{\lambda_\theta\lambda_z}\left(\frac{\partial^2u_1}{\partial\xi^2}-2\frac{\partial^2u_1}{\partial\xi\partial\eta}+\frac{\partial^2u_1}{\partial\eta^2}\right).
\end{aligned} \tag{22}$$

For the solution of the set (21), we shall eliminate $w_1$ between these equations and obtain

$$\begin{aligned}
&-\left(\frac{2c_0}{\lambda_\theta}-\frac{\beta_1}{c_0}\right)\left(\frac{\partial^2u_1}{\partial\xi^2}+\frac{\partial^2u_1}{\partial\eta^2}\right)\\
&+2\left(\frac{2c_0}{\lambda_\theta}+\frac{\beta_1}{c_0}\right)\frac{\partial^2u_1}{\partial\xi\partial\eta}=0.
\end{aligned} \tag{23}$$

It can be shown that, in the longwave limit, $c_0$ is the phase velocity and may be given by

$$c_0^2=\frac{\lambda_\theta\beta_1}{2}. \tag{24}$$

Then, (23) reduces to

$$\frac{\partial^2u_1}{\partial\xi\partial\eta}=0. \tag{25}$$

The solution of this equation yields

$$u_1=U(\xi)+V(\eta), \tag{26}$$

where $U(\xi)$ and $V(\eta)$ are two unknown functions whose governing equations will be obtained later.

Introducing (26) into (21) we have

$$w_1=\frac{2c_0}{\lambda_\theta}(U-V),\quad p_1=\frac{2c_0^2}{\lambda_\theta}(U+V). \tag{27}$$

To obtain the solution to $O(\varepsilon^2)$ order equations, we introduce the solution given in (26) and (27) into (22), which yields

$$\begin{aligned}
&\frac{2c_0}{\lambda_\theta}\left(-\frac{\partial u_2}{\partial\xi}+\frac{\partial u_2}{\partial\eta}\right)+\left(\frac{\partial w_2}{\partial\xi}+\frac{\partial w_2}{\partial\eta}\right)=F(\xi,\eta),\\
&\frac{2c_0}{\lambda_\theta}\left(\frac{\partial u_2}{\partial\xi}+\frac{\partial u_2}{\partial\eta}\right)+\left(-\frac{\partial w_2}{\partial\xi}+\frac{\partial w_2}{\partial\eta}\right)=G(\xi,\eta),
\end{aligned} \tag{28}$$

where the functions $F(\xi,\eta)$ and $G(\xi,\eta)$ are defined by

$$\begin{aligned}
F(\xi,\eta)=&\left(\frac{2}{\lambda_\theta}c_{A1}\frac{\mathrm{d}U}{\mathrm{d}\xi}-\frac{6c_0}{\lambda_\theta^2}U\frac{\mathrm{d}U}{\mathrm{d}\xi}\right)+\left(-\frac{2}{\lambda_\theta}c_{B1}\frac{\mathrm{d}V}{\mathrm{d}\eta}+\frac{6c_0}{\lambda_\theta^2}V\frac{\mathrm{d}V}{\mathrm{d}\eta}\right)+\left(-\frac{4c_0}{\lambda_\theta}\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}+\frac{2c_0}{\lambda_\theta^2}V\right)\frac{\mathrm{d}U}{\mathrm{d}\xi}\\
&+\left(\frac{4c_0}{\lambda_\theta}\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}-\frac{2c_0}{\lambda_\theta^2}U\right)\frac{\mathrm{d}V}{\mathrm{d}\eta},\\
G(\xi,\eta)=&\left[\left(\frac{\alpha_0}{c_0}-\frac{mc_0}{\lambda_\theta\lambda_z}\right)\frac{\mathrm{d}^3U}{\mathrm{d}\xi^3}-\left(\frac{2\beta_2}{c_0}+\frac{4c_0}{\lambda_\theta^2}\right)U\frac{\mathrm{d}U}{\mathrm{d}\xi}+\frac{2c_{A1}}{\lambda_\theta}\frac{\mathrm{d}U}{\mathrm{d}\xi}\right]\\
&+\left[\left(\frac{\alpha_0}{c_0}-\frac{mc_0}{\lambda_\theta\lambda_z}\right)\frac{\mathrm{d}^3V}{\mathrm{d}\eta^3}-\left(\frac{2\beta_2}{c_0}+\frac{4c_0}{\lambda_\theta^2}\right)V\frac{\mathrm{d}V}{\mathrm{d}\eta}+\frac{2c_{B1}}{\lambda_\theta}\frac{\mathrm{d}V}{\mathrm{d}\eta}\right]\\
&+\left[-\frac{4c_0}{\lambda_\theta}\frac{\mathrm{d}\theta_0}{\mathrm{d}\eta}-\left(\frac{2\beta_2}{c_0}-\frac{4c_0}{\lambda_\theta^2}\right)V\right]\frac{\mathrm{d}U}{\mathrm{d}\xi}\left[-\frac{4c_0}{\lambda_\theta}\frac{\mathrm{d}\phi_0}{\mathrm{d}\xi}-\left(\frac{2\beta_2}{c_0}-\frac{4c_0}{\lambda_\theta^2}\right)U\right]\frac{\mathrm{d}V}{\mathrm{d}\eta}.
\end{aligned} \tag{29}$$

The solution of the homogeneous equation given in (28) yields the result similar to the $O(\varepsilon)$ case. For our future use, we need the particular solution of (28). The particular integral of (28) gives

$$u_2 = \frac{\lambda_\theta}{8c_0}\left\{\int^{\eta}[F(\xi,\eta')+G(\xi,\eta')]\mathrm{d}\eta' + \int^{\xi}[G(\xi',\eta)-F(\xi',\eta)]\mathrm{d}\xi'\right\},$$
$$w_2 = \frac{1}{4}\left\{\int^{\eta}[F(\xi,\eta')+G(\xi,\eta')]\mathrm{d}\eta' - \int^{\xi}[G(\xi',\eta)-F(\xi',\eta)]\mathrm{d}\xi'\right\}. \quad (30)$$

Introducing the expressions of $F(\xi,\eta)$ and $G(\xi,\eta)$ into (30) we have

$$u_2 = \frac{\lambda_\theta}{8c_0}\Big[A_1(\xi)\eta + B_2(\eta)\xi + A_2(\xi) + B_1(\eta) - C_1(\eta)\frac{\mathrm{d}U}{\mathrm{d}\xi} - C_2(\xi)\frac{\mathrm{d}V}{\mathrm{d}\eta} + 4\left(\frac{c_0}{\lambda_\theta^2} - \frac{\beta_2}{c_0}\right)U(\xi)V(\eta)\Big], \quad (31)$$
$$w_2 = \frac{1}{4}\Big[A_1(\xi)\eta - B_2(\eta)\xi + B_1(\eta) - A_2(\xi) - C_1(\eta)\frac{\mathrm{d}U}{\mathrm{d}\xi} + C_2(\xi)\frac{\mathrm{d}V}{\mathrm{d}\eta}\Big],$$

where various functions appearing in (31) are defined by

$$A_1(\xi) = \frac{4c_{A1}}{\lambda_\theta}\frac{\mathrm{d}U}{\mathrm{d}\xi} - \left(\frac{2\beta_2}{c_0} + \frac{10c_0}{\lambda_\theta^2}\right)U\frac{\mathrm{d}U}{\mathrm{d}\xi} + \left(\frac{\alpha_0}{c_0} - \frac{mc_0}{\lambda_\theta\lambda_z}\right)\frac{\mathrm{d}^3U}{\mathrm{d}\xi^3},$$
$$B_1(\eta) = \left(\frac{\alpha_0}{c_0} - \frac{mc_0}{\lambda_\theta\lambda_z}\right)\frac{\mathrm{d}^2V}{\mathrm{d}\eta^2} + \left(\frac{c_0}{\lambda_\theta^2} - \frac{\beta_2}{c_0}\right)V^2,$$
$$C_1(\eta) = \frac{8c_0}{\lambda_\theta}\theta_0 + \left(\frac{2\beta_2}{c_0} - \frac{6c_0}{\lambda_\theta^2}\right)\int^{\eta}V(\eta')\mathrm{d}\eta',$$
$$A_2(\xi) = \left(\frac{\alpha_0}{c_0} - \frac{mc_0}{\lambda_\theta\lambda_z}\right)\frac{\mathrm{d}^2U}{\mathrm{d}\xi^2} + \left(\frac{c_0}{\lambda_\theta^2} - \frac{\beta_2}{c_0}\right)U^2,$$
$$B_2(\eta) = \frac{4c_{B1}}{\lambda_\theta}\frac{\mathrm{d}V}{\mathrm{d}\eta} - \left(\frac{2\beta_2}{c_0} + \frac{10c_0}{\lambda_\theta^2}\right)V\frac{\mathrm{d}V}{\mathrm{d}\eta} + \left(\frac{\alpha_0}{c_0} - \frac{mc_0}{\lambda_\theta\lambda_z}\right)\frac{\mathrm{d}^3V}{\mathrm{d}\eta^3},$$
$$C_2(\xi) = \frac{8c_0}{\lambda_\theta}\phi_0 + \left(\frac{2\beta_2}{c_0} - \frac{6c_0}{\lambda_\theta^2}\right)\int^{\xi}U(\xi')\mathrm{d}\xi'. \quad (32)$$

Since we are concerned with the localized solutions for the field quantities $U(\xi)$ and $V(\eta)$, these functions must be bounded. As can be seen from (31) as $\xi,\eta \to \pm\infty$, the solution becomes unbounded unless

$$A_1(\xi) = 0, \quad B_2(\eta) = 0, \quad (33)$$

which yield the following ordinary differential equations for $U$ and $V$:

$$c_{A1}\frac{\mathrm{d}U}{\mathrm{d}\xi} - \mu_1 U\frac{\mathrm{d}U}{\mathrm{d}\xi} - \mu_2\frac{\mathrm{d}^3U}{\mathrm{d}\xi^3} = 0, \quad (34)$$

$$c_{B1}\frac{\mathrm{d}V}{\mathrm{d}\eta} - \mu_1 V\frac{\mathrm{d}V}{\mathrm{d}\eta} - \mu_2\frac{\mathrm{d}^3V}{\mathrm{d}\eta^3} = 0, \quad (35)$$

where the coefficients $\mu_1$ and $\mu_2$ are defined by

$$\mu_1 = \left(\frac{\lambda_\theta\beta_2}{2c_0} + \frac{5c_0}{2\lambda_\theta}\right), \; \mu_2 = \frac{\lambda_\theta}{4}\left(\frac{mc_0}{\lambda_\theta\lambda_z} - \frac{\alpha_0}{c_0}\right). \quad (36)$$

As a matter of fact, (34) and (35) are the travelling wave form of the Korteweg-de Vries equations given by

$$\frac{\partial U}{\partial\tau} + \mu_1 U\frac{\partial U}{\partial z_1} + \mu_2\frac{\partial U}{\partial z_1^3} = 0, \quad (37)$$

$$-\frac{\partial V}{\partial\tau} + \mu_1 V\frac{\partial V}{\partial z_2} + \mu_2\frac{\partial^3 V}{\partial z_2^3} = 0, \quad (38)$$

where we have set

$$\tau = \varepsilon^{3/2}t, \quad z_1 = \varepsilon^{1/2}(z - c_0 t),$$
$$z_2 = \varepsilon^{1/2}(z + c_0 t), \quad \xi - \varepsilon\theta_0(\eta) = z_1 - c_{A1}t, \quad (39)$$
$$\eta - \varepsilon\phi_0(\xi) = z_2 + c_{B1}t.$$

Due to the existence of the functions $\theta_0(\eta)$ and $\phi_0(\xi)$, the wave trajectories are not straight lines in the $z_\alpha,\tau$ space, they are rather some curves. This is the result of the head-on collision of waves.

Here we notice that $\theta_0(\eta)$ and $\phi(\xi)$ are some unknown functions to be determined. Without losing the generality of the problem we may set the coefficient functions $C_1(\eta)$ and $C_2(\xi)$ equal to zero, which yields to

$$\frac{8c_0}{\lambda_\theta}\theta_0(\eta) + \left(\frac{2\beta_2}{c_0} - \frac{6c_0}{\lambda_\theta^2}\right)\int^{\eta}V(\eta')\mathrm{d}\eta' = 0, \quad (40)$$

$$\frac{8c_0}{\lambda_\theta}\phi_0(\xi)+\left(\frac{2\beta_2}{c_0}-\frac{6c_0}{\lambda_\theta^2}\right)\int^{\xi}U(\xi')\mathrm{d}\xi'=0. \quad (41)$$

These equations make it possible to determine the unknown functions $\theta_0(\eta)$ and $\phi_0(\xi)$, provided that $U(\xi)$ and $V(\eta)$ are known. As a matter of fact, $U(\xi)$ and $V(\eta)$ can be obtained from the solution of the ordinary differential equations (34) and (35). These differential equations assume a solution of the following form:

$$U=a\,\mathrm{sech}^2\lambda\xi,\quad V=b\,\mathrm{sech}^2\mu\eta, \quad (42)$$

where $a$ and $b$ are the amplitudes of the solitary waves and $\lambda$ and $\mu$ are two constants to be determined from the solution of (34) and (35). Introducing (42) into (34) and (35) we obtain

$$\lambda=\left(\frac{\mu_1 a}{12\mu_2}\right)^{1/2},\quad \mu=\left(\frac{\mu_1 b}{12\mu_2}\right)^{1/2},$$
$$c_{A1}=\frac{\mu_1 a}{3},\quad c_{B1}=\frac{\mu_1 b}{3}. \quad (43)$$

Having obtained the solution for $U(\xi)$ and $V(\eta)$, through the use of (40) and (41) one can determine the phase variables $\theta_0(\eta)$ and $\phi_0(\xi)$ as

$$\theta_0(\eta)=\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)\left(\frac{12\mu_2 b}{\mu_1}\right)^{1/2}\tanh\mu\eta,$$
$$\phi_0(\xi)=\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)\left(\frac{12\mu_2 a}{\mu_1}\right)^{1/2}\tanh\lambda\xi. \quad (44)$$

Hence, up to $O(\varepsilon^{3/2})$, the trajectories of the two solitary waves for weak interaction are

$$\xi=\varepsilon^{1/2}(z-c_0t-\varepsilon c_{A1}t)+\varepsilon\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)\left(\frac{12\mu_2 b}{\mu_1}\right)^{1/2}\tanh\mu\eta+O(\varepsilon^{3/2}),$$
$$\eta=\varepsilon^{1/2}(z+c_0t+\varepsilon c_{B1}t)+\varepsilon\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)\left(\frac{12\mu_2 a}{\mu_1}\right)\tanh\lambda\xi+O(\varepsilon^{3/2}). \quad (45)$$

To obtain the phase shifts after the head-to-head collision of the solitary waves, we assume that the solitary waves, characterized by $A$ and $B$, are asymptotically far from each other at the initial time ($t=-\infty$). The solitary wave $A$ is at $\xi=0$, $\eta=-\infty$, and the solitary wave $B$ is at $\eta=0$, $\xi=\infty$, respectively. After the head-to-head collision ($t=\infty$), the solitary wave $B$ is far to the right of the solitary wave $A$, i. e., the solitary wave $A$ is at $\xi=0$, $\eta=\infty$, and the solitary wave $B$ is at $\eta=0$, $\xi=-\infty$. For these limiting cases the trajectories (45) become

$$z=(c_0+\varepsilon c_{A1})t\pm\varepsilon^{1/2}\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)\left(\frac{12\mu_2 b}{\mu_1}\right)^{1/2},$$
$$z=(c_0+\varepsilon c_{B1})t\pm\varepsilon^{1/2}\left(\frac{3}{4\lambda_\theta}-\frac{\beta_2}{2\beta_1}\right)\left(\frac{12\mu_2 a}{\mu_1}\right)^{1/2}. \quad (46)$$

These equations are parallel straight lines which represent the asymptotes of the corresponding trajectories.

Using (45), we obtain the corresponding phase shifts $\Delta_A$ and $\Delta_B$ as follows:

$$\Delta_A=\varepsilon^{1/2}(z-c_At)|_{\xi=0,\eta=\infty}-\varepsilon^{1/2}(z-c_At)|_{\xi=0,\eta=-\infty}=\varepsilon\left(\frac{3}{2\lambda_\theta}-\frac{\beta_2}{\beta_1}\right)\left(\frac{12\mu_2 b}{\mu_1}\right)^{1/2}, \quad (47)$$

$$\Delta_B=\varepsilon^{1/2}(z+c_Bt)|_{\eta=0,\xi=-\infty}-\varepsilon^{1/2}(z+c_Bt)|_{\eta=0,\xi=\infty}=\varepsilon\left(\frac{3}{2\lambda_\theta}-\frac{\beta_2}{\beta_1}\right)\left(\frac{12\mu_2 a}{\mu_1}\right)^{1/2}. \quad (48)$$

## 4. Numerical Results and Discussion

For the numerical analysis we need the constitutive equations for the elastic tube material. For that purpose we shall use the constitutive equations proposed by Demiray [10] for soft biological tissues as

$$\Sigma=\frac{1}{2\alpha}\{\exp[\alpha(I_1-3)]-1\}, \quad (49)$$

where $\alpha$ is a material constant and $I_1$ is the first invariant of the Finger deformation tensor and defined by $I_1=\lambda_z^2+\lambda_\theta^2+1/\lambda_z^2\lambda_\theta^2$. Introducing (49) into (23), the coefficients $\alpha_0$, $\beta_0$, $\beta_1$ and $\beta_2$ are obtained as:

$$\alpha_0=\left(\lambda_z^2-\frac{1}{\lambda_\theta^2\lambda_z^2}\right)F(\lambda_\theta,\lambda_z),$$
$$\beta_0=\left(\frac{1}{\lambda_z}-\frac{1}{\lambda_\theta^4\lambda_z^3}\right)F(\lambda_\theta,\lambda_z),$$
$$\beta_1=\left[\frac{4}{\lambda_\theta^5\lambda_z^3}+2\frac{\alpha}{\lambda_\theta\lambda_z}\left(\lambda_\theta-\frac{1}{\lambda_\theta^3\lambda_z^2}\right)^2\right]F(\lambda_\theta,\lambda_z),$$
$$\beta_2=\left[-\frac{10}{\lambda_\theta^6\lambda_z^3}+\frac{\alpha}{\lambda_\theta\lambda_z}\left(5+\frac{7}{\lambda_\theta^4\lambda_z^2}\right)\left(\lambda_\theta-\frac{1}{\lambda_\theta^3\lambda_z^2}\right)+2\frac{\alpha^2}{\lambda_\theta\lambda_z}\left(\lambda_\theta-\frac{1}{\lambda_\theta^3\lambda_z^2}\right)^3\right]F(\lambda_\theta,\lambda_z), \quad (50)$$

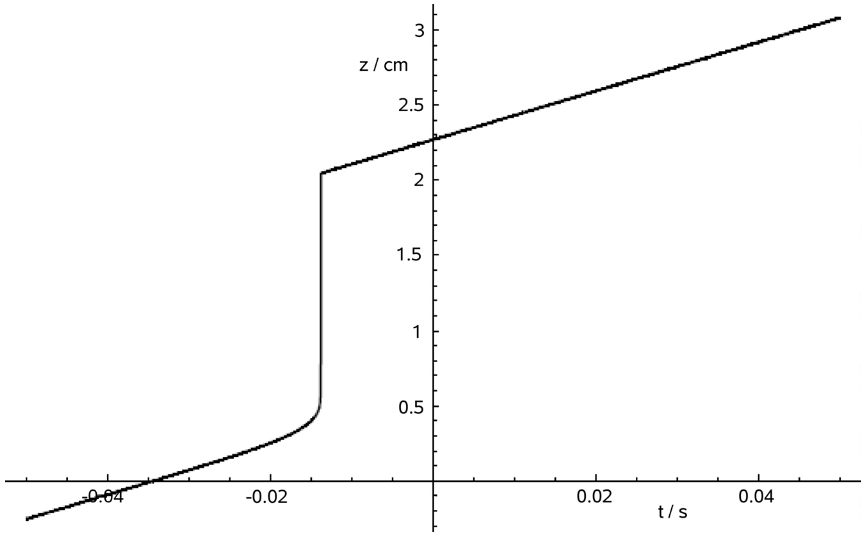


Fig. 1. Variation of a trajectory for $\xi_0 = 1.0$.

where the function $F(\lambda_\theta,\lambda_z)$ is defined by

$$F(\lambda_\theta,\lambda_z) = \exp\left[\alpha\left(\lambda_\theta^2+\lambda_z^2+\frac{1}{\lambda_\theta^2\lambda_z^2}-3\right)\right]. \quad (51)$$

This theoretical model was compared by Demiray [11] with the experimental measurements by Simon et al. [12] on a canine abdominal artery with the characteristics $R_i = 0.31$ cm, $R_0 = 0.38$ cm and $\lambda_z = 1.53$, and the value of the material constant $\alpha$ was found to be $\alpha = 1.948$. Using this value of the material constant and 1.6 for $\lambda_\theta = \lambda_z$, the coefficients $\alpha_0$, $\beta_1$, $\beta_2$, $\mu_1$, $\mu_2$ were calculated. The result is

$$\frac{\alpha_0}{\beta_1} = 0.266, \quad \frac{\beta_2}{\beta_1} = 3.348,$$
$$\mu_1 = 4.911, \quad \mu_2 = -0.0363.$$

Setting $\varepsilon = 0.5$ and the wave amplitudes $a = -1$, $b = -2$, the phase functions $\xi$ and $\eta$ take the forms

$$\xi = 0.707(z-16.21t) - 0.606\tanh(4.748\eta), \quad (52)$$

$$\eta = 0.707(z+17.21t) - 0.429\tanh(3.358\xi). \quad (53)$$

As is seen from these expressions the trajectories are not straight lines anymore, they are rather some curves in the $(z,t)$ plane.

Setting $\xi = \xi_0$ in (52) and (53) we obtain

$$\xi_0 = 0.707(z-16.21t) - 0.606\tanh(4.748\eta),$$
$$\eta = 0.707(z+17.21t) - 0.429\tanh(3.358\xi_0). \quad (54)$$

This gives the equation of the trajectory for $\xi = \xi_0$. For large values of $\eta$ $(\eta \to \pm\infty)$, (52) becomes

$$\xi_0 = 0.707z - 11.46t \pm 0.606. \quad (55)$$

These are the equations of the asymptotes, which are parallel lines of the trajectory. Similarly, the equation of the trajectory for $\eta = \eta_0$ may be obtained as

$$\xi = 0.707(z-16.21t) - 0.606\tanh(4.748\eta_0),$$
$$\eta_0 = 0.707(z+17.21t) - 0.429\tanh(3.358\xi). \quad (56)$$

For large values of $\xi$ $(\xi \to \pm\infty)$, (56) becomes

$$\eta_0 = 0.707z + 12.167t \pm 0.429. \quad (57)$$

The trajectories are plotted numerically and the results are depicted on Figs. 1 and 2 for $\xi_0 = 1.0$ and $\eta_0 = 1.0$. As can be seen from these figures, for large values of $\xi$ and $\eta$ the trajectories are two parallel lines, but during the collision it becomes a sharp curve that connects these two lines. This means that, when the collision process is completed, there will be a phase shift.

## 5. Conclusion

By use of the extended PLK perturbation method, the head-to-head collision of two solitary waves in an artery is investigated. The result obtained shows that, up to $O(k^4)$, the head-to-head collision of two blood solitons is elastic and the solitons preserve their original properties after the collision. The leading order analytical phase shifts of a head-to-head

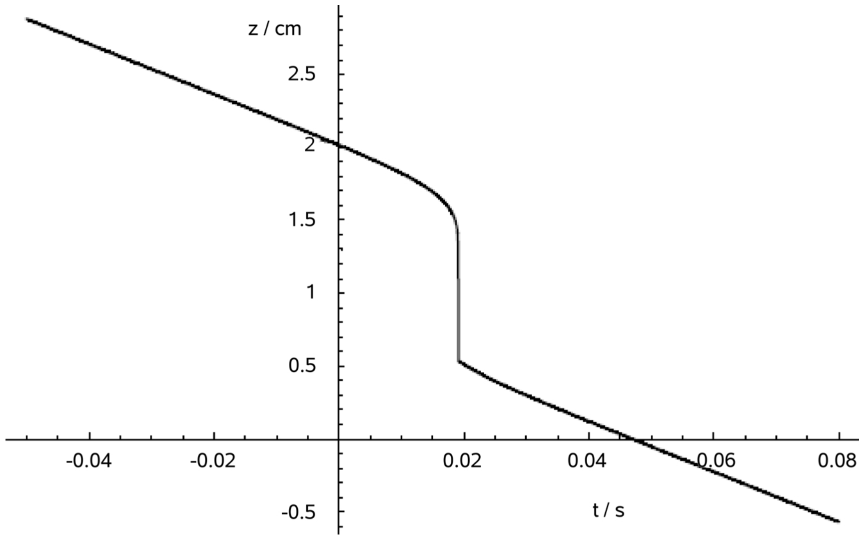


Fig. 2. Variation of a trajectory for $\eta_0 = 1.0$.

collision between two solitary waves are explicitly derived. The higher-order corrections may give some secondary structures in the collision event, especially for the large amplitude case.

*Acknowledgements*

This work was partially supported by the Turkish Academy of Sciences (TÜBA).

[1] D. A. Mc Donald, Blood Flow in Arteries, 2nd ed., Edward Arnold, London 1974.
[2] Y. Hashizume, J. Phys. Soc. Jpn. **54**, 3305 (1985).
[3] S. Yomosa, J. Phys. Soc. Jpn. **56**, 506 (1987).
[4] J. F. Paquerot and M. Remoissenet, Phys. Lett. A **194**, 77 (1994).
[5] H. Demiray, Int. J. Eng. Sci. **42**, 203 (2004).
[6] H. Demiray, Z. Naturforsch. **61a**, 641 (2006).
[7] C. H. Su and R. M. Mirie, J. Fluid Mech. **98**, 509 (1980).
[8] A. Jeffrey and T. Kawahara, Asymptotic Methods in Nonlinear Wave Theory, Pitman, London 1982.
[9] G. Rudinger, J. Appl. Mech. **37**, 34 (1970).
[10] H. Demiray, J. Biomech. **5**, 309 (1972).
[11] H. Demiray, Bull. Math. Biol. **38**, 701 (1976).
[12] B. R. Simon, A. S. Kobayashi, D. E. Stradness, and C. A. Wiederhielm, Circulation Res. **30**, 491 (1972).